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TAUBERIAN THEOREM OF ERDŐS REVISITED

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Dedicated to the memory of Paul Erdős

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In connection with the elementary proof of the prime number theorem, Erdős obtained a striking quadratic Tauberian theorem for sequences. Somewhat later, Siegel indicated in a letter how a powerful "fundamental relation" could be used to simplify the difficult combinatorial proof. Here the author presents his version of the (unpublished) Erdős–Siegel proof. Related Tauberian results by the author are described.

1. Introduction

At the time of the elementary proof of the prime number theorem, Erdős obtained the following nonlinear Tauberian result [3].

Theorem 1.1. *Let $a_n \geq 0$ and $s_n = \sum_{k=1}^{n} a_k$, $n=1,2,\dots$. Suppose that*

$$\Sigma_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} a_k(s_{n-k}+k) = n^2 + \mathcal{O}(n) \tag{1.1}$$

(where $s_0 = 0$). Then

$$r_n \stackrel{\text{def}}{=} s_n - n = \mathcal{O}(1). \tag{1.2}$$

The important thing here is the strength of the remainder estimate. Starting with Selberg's asymptotic formula for the primes p_i in the sharp form

$$\sum_{p_i \leq z} (\log p_i)^2 + \sum_{p_i p_j \leq z} \log p_i \log p_j = 2z \log z + \mathcal{O}(z),$$

Mathematics Subject Classification (2000): 40E05; 11A41 see [10], Erdős showed in [3] how Theorem 1.1 leads to the prime number theorem with no further information about the real numbers p_i than that $1<p_1<p_2<\cdots$.

In the Fall of 1948, Erdős gave several lectures in Amsterdam on the elementary proof of the prime number theorem by Selberg [10] and himself [2]. These lectures resulted in the first published version of that proof (Van der Corput [1]). At that time, I was one of the two junior mathematicians working at the newly founded Mathematical Center in Amsterdam of which Van der Corput was the director. Since I had already published on Tauberian theorems, Erdős involved me in the verification of his very complicated combinatorial proof for Theorem 1.1.

Somewhat later (Spring 1950 at Purdue University), Erdős showed me a letter from C.L. Siegel [12] which sketched how one could simplify the final part of the proof. Siegel's main tool was an ingenious "fundamental relation" obtained by combining (1.1) and (1.2) (see Section 2.2). Several years later, H.N. Shapiro [11] published a quite different simplification of Erdős's proof. Erdős considered the Tauberian theorem one of his main accomplishments, cf. [4]. Since the integrated Erdős–Siegel proof below is more transparent than the proofs which have appeared earlier, I think that it is worth publishing.

The present proof proceeds in a number of steps which parallel those in Erdős [3] up to Step 5 below. However, Siegel's fundamental relation is brought in already for Step 2. The "end game" in Section 3.3 is based on Siegel's unpublished sketch [12]. In the course of the proof I will also establish the following supplement to Theorem 1.1 which Erdős had stated without proof [3].

Theorem 1.2. *With the notations of Theorem 1.1, suppose that $a_n \geq 0$ and*

$$\Sigma_n = \sum_{k=1}^{n} a_k(s_{n-k}+k) = n^2 + \mathcal{O}(n^{1+\gamma}) \quad \text{where } \ 0<\gamma<1. \tag{1.3}$$

Then

$$r_n = \mathcal{O}(n^{\gamma}). \tag{1.4}$$

Remark 1.3. In his Supplementary Note at the end of [3], Erdős observed that the condition

$$B_n \stackrel{\text{def}}{=} \sum_{k=1}^{n} a_k s_{n-k} - n^2/2 = \mathcal{O}(n) \tag{1.5}$$

for nonnegative a_n implies $r_n = o(n)$ but not $o(n^{\frac{1}{2}})$. Several improvements have been published. For references see Korevaar [8], where Laplace transformation is used to show that (1.5) implies $r_n = \mathcal{O}(n^{\frac{3}{5}})$. That estimate holds also under the related mean-square condition $\sum_{k=1}^{n} B_k^2 = \mathcal{O}(n^3)$ and then it is optimal. Under condition (1.5) the optimal remainder might be $r_n = \mathcal{O}(n^{\frac{1}{2}})$.

More natural Tauberian problems of convolution type involve order conditions on

$$b_n = \sum_{k=1}^{n}(a_k a_{n-k} - 1) \quad \text{or} \quad b_n = \sum_{k_1+\cdots+k_p=n}(a_{k_1}\cdots a_{k_p} - 1).$$

The author has recently treated such problems with the aid of power series and complex analysis [9].

2. Fundamental relation and proof of Theorem 1.2

It is assumed throughout this chapter that the hypotheses of Theorem 1.2 are satisfied.

2.1. Step 1. *Suppose one has (1.3) or just*

$$\varSigma_n = n^2 + o(n^2) \quad \text{as } n \to \infty. \tag{2.1}$$

Then

$$r_n = o(n). \tag{2.2}$$

Relation (2.2) was also Erdős's first step under hypothesis (1.1), but under the weaker condition (2.1) the proof is longer.

Proof. One has to show that $\alpha = \underline{\lim}\, s_n/n$ is equal to $\beta = \overline{\lim}\, s_n/n$.

By the definition of $\varSigma_n$ and partial summation

$$\varSigma_n = \sum_{k=1}^{n} a_k(s_{n-k} + k) \geq \sum_{1}^{n} k a_k = n s_n - \sum_{1}^{n-1} s_k. \tag{2.3}$$

We first prove that $\beta < \infty$. Assuming for the moment that $\beta = \infty$, one focuses on large n for which $T_n = s_n/n \geq s_k/k$ for all $k \leq n$. Then by (2.3) and (2.1)

$$n^2 T_n \leq \sum_{1}^{n-1} k T_n + \varSigma_n \leq \frac{1}{2} n^2 T_n + \{1 + o(1)\} n^2,$$

so that $T_n \leq 2 + o(1)$. Thus $\beta \leq 2$ which contradicts the assumption $\beta = \infty$.

If $\alpha \geq 1$ then $s_k \geq k - o(n)$ for $k \leq n$, hence

$$n^2 + o(n^2) = \sum_1^n a_k(s_{n-k} + k) \geq [n - o(n)] \sum_1^n a_k,$$

so that $\beta \leq 1$ and therefore $\alpha = \beta$. Similarly $\beta \leq 1$ implies $\alpha \geq 1$ and $\alpha = \beta$. Thus we may assume for the remainder of the proof that

$$\alpha < 1 < \beta. \tag{2.4}$$

Now observe that $s_{n-k} + k \geq \alpha n + (1-\alpha)k - o(n)$, so that by (2.3)

$$\Sigma_n \geq [\alpha n - o(n)]s_n + (1-\alpha)\left(ns_n - \sum_1^{n-1} s_k\right).$$

Choosing n such that $s_n = \beta n + o(n)$ one concludes that

$$n^2 + o(n^2) \geq [\alpha\beta + (1-\alpha)\beta/2]n^2 - o(n^2)$$

which implies $\alpha\beta < 1$.

We finally take n such that $s_n = \alpha n + o(n)$. Then for small $\rho > 0$ and $k \leq \rho n$,

$$s_{n-k} + k \leq \alpha n + k + o(n) \leq (\alpha + \rho)n + o(n) \leq (\beta - \delta)n + o(n)$$

with $\delta > 0$. Also, for $k > \rho n$,

$$s_{n-k} + k \leq \beta n + (1-\beta)k + o(n) \leq \beta n - (\beta - 1)\rho n + o(n) \leq (\beta - \delta)n + o(n),$$

provided δ had been chosen small enough. It follows that

$$n^2 - o(n^2) \leq \Sigma_n \leq (\beta - \delta)ns_n + o(n^2) \leq (\beta - \delta)\alpha n^2 + o(n^2)$$

which would imply $\alpha\beta > 1$.

This contradiction proves that (2.4) is false so that $\alpha = 1$ or $\beta = 1$, and by the preceding it follows that indeed $\alpha = \beta = 1$.

Relation (2.2) could also be derived with the aid of power series. Under hypothesis (1.3), the function $y = \sum a_k e^{-kx}$ satisfies a differential equation

$$y' - y^2 = -2x^{-2} + \mathcal{O}(x^{-(1+\gamma)}) \quad \text{as} \quad x \downarrow 0.$$

Since $a_k \geq 0$ one can deduce from this Riccati equation that $y = x^{-1} + \mathcal{O}(x^{-\gamma}) \sim x^{-1}$ as $x \downarrow 0$. Hence by a Tauberian theorem of Hardy and Littlewood (1913, see [6] Theorems 95–97), one has $s_n \sim n$. More precisely, the Tauberian remainder theory of Freud and Korevaar [5], [7] would give

$r_n = \mathcal{O}(n/\log n)$, but this seems to be the best that real Tauberian methods are capable of.

2.2. Siegel's fundamental relation. Replacing n by q and substituting $s_k = k + r_k$, equations (1.2) and (1.3) may be rewritten as

$$\text{(2.5)} \qquad \sum_{0 \le k < q} a_{q-k} = q + r_q, \qquad \sum_{0 \le k < q} a_{q-k} \cdot (-r_k) = q r_q + \mathcal{O}(q^{1+\gamma}).$$

We now multiply the first relation by $1 - R/D$ and the second one by η/D. Adding, one obtains the following

Fundamental relation. *For $0 < D < R$ and $\eta = \pm 1$,*

$$\text{(2.6)} \sum_{k<q} a_{q-k}\left(1 - \frac{R + \eta r_k}{D}\right) = q\left(1 - \frac{R - \eta r_q}{D}\right) + \left(1 - \frac{R}{D}\right) r_q + \mathcal{O}\left(\frac{q^{1+\gamma}}{D}\right).$$

This relation will be applied several times with q large, $n \ge q$ and

$$\text{(2.7)} \qquad R = R_n, \quad \text{where } \frac{R_n}{n^\gamma} \stackrel{\text{def}}{=} \max_{1 \le k \le n} \frac{|r_k|}{k^\gamma}.$$

In the proofs it may then be assumed that

(2.8) the nondecreasing sequence $\{R_n/n^\gamma\}$ is unbounded,

or there would be nothing to prove.

2.3. Proof of Theorem 1.2. We are ready for

Step 2. *Let (1.3) be satisfied where $\gamma > 0$. Then*

$$\text{(2.9)} \qquad r_n = \mathcal{O}(n^\gamma).$$

Proof. Suppose that (2.9) is false, so that $R_n/n^\gamma \nearrow \infty$ (2.8). We will focus on a sequence of $n \to \infty$ for which $R_n = |r_n|$ (the latter may be ensured by taking n such that $R_n/n^\gamma > R_{n-1}/(n-1)^\gamma$). Observe that

$$\text{(2.10)} \qquad |r_k| \le (k/n)^\gamma R_n < 2^{-\gamma} R_n, \quad \forall\, k < n/2.$$

We now apply the fundamental relation with $q = n$, $R = R_n$, $D = (1 - 2^{-\gamma})R$, $\eta = \pm 1 = R/r_n$. The result is

$$\sum_{k < \frac{1}{2}n} a_{n-k}\left(1 - \frac{R_n \pm r_k}{(1 - 2^{-\gamma})R_n}\right) + \sum_{k \ge \frac{1}{2}n} a_{n-k}\left(1 - \frac{R_n \pm r_k}{(1 - 2^{-\gamma})R_n}\right)$$
$$= n - \frac{r_n}{2^\gamma - 1} + \mathcal{O}\left(\frac{n^{1+\gamma}}{R_n}\right).$$

Hence by (2.10), (2.2) in Step 1 and (2.8),

$$\sum_{k<\frac{1}{2}n} a_{n-k} \cdot \text{neg} + \sum_{k\geq\frac{1}{2}n} a_{n-k}(1+\text{neg}) = n - o(n).$$

Here "neg" stands for numbers ≤ 0. It follows that

$$s_{\frac{1}{2}n} = \sum_{k\geq\frac{1}{2}n} a_{n-k} \geq n - o(n), \quad (s_\nu = s_{[\nu]}).$$

However, by Step 1 the left-hand side is asymptotic to $n/2$ as $n \to \infty$. This contradiction establishes (2.9) and hence Theorem 1.2.

3. Proof of Theorem 1.1

It is assumed throughout this chapter that the hypotheses of Theorem 1.1 are satisfied. We then have the fundamental relation with $\gamma = 0$.

3.1. Initial reduction by Erdős.

Step 3. *The sequence $\{a_k\}$ is bounded and for the proof of Theorem 1.1, it may and will be assumed that for a suitable arbitrarily small positive number ε,*

(3.1) $$a_k < 2 + \varepsilon \quad \text{for all} \quad k \geq k_0.$$

Proof. (i) Subtracting Σ_n in (1.1) from Σ_{n+1}, one finds that $(n+1)a_{n+1} \leq \Sigma_{n+1} - \Sigma_n = 2n + \mathcal{O}(n)$, so that (say)

(3.2) $$a_k \leq c.$$

(ii) Suppose now that Theorem 1.1 has been proved under the additional condition (3.1). Then the general case may be handled as follows. Define

$$\tilde{a}_k = \frac{1}{p} \sum_{m=(k-1)p+1}^{kp} a_m.$$

Then by (1.1)

$$\begin{aligned}(k-1)p^2\tilde{a}_k &= (k-1)p\{a_{(k-1)p+1} + \cdots + a_{kp}\} \\ &\leq \Sigma_{kp} - \Sigma_{(k-1)p} = (2k-1)p^2 + \mathcal{O}(kp),\end{aligned}$$

hence for any given $\varepsilon > 0$ we can choose a fixed $p = p(\varepsilon)$ so large that the numbers $\tilde{a}_k$ satisfy (3.1). Furthermore, since $a_k \leq c$,

$$\sum_{k=1}^{n} \tilde{a}_k(\tilde{a}_1 + \cdots + \tilde{a}_{n-k} + k)$$
$$= \frac{1}{p^2} \sum_{m=1}^{pn} a_m(a_1 + \cdots + a_{pn-m} + m) + \mathcal{O}(n) = n^2 + \mathcal{O}(n).$$

Thus by our supposition, $\sum_{k=1}^{n} \tilde{a}_k = n + \mathcal{O}(1)$. This implies (1.2) by the definition of $\tilde{a}_k$ and the boundedness of the a_k's.

3.2. More delicate steps. In accordance with (2.8) (now with $\gamma = 0$) we proceed under the assumption that r_n is unbounded. As before we let n run through a subsequence of the positive integers such that

$$|r_n| = R_n = \max_{k \leq n} |r_k|; \quad \text{supposition} : R_n \nearrow \infty. \tag{3.3}$$

For definiteness we assume that $r_n > 0$ (see Remark 3.1 for the case $r_n < 0$). One now sets

$$r_n = R_n = R \quad \text{and} \quad \log R = \rho. \tag{3.4}$$

As in Erdős's article, indices $k \leq n$ for which r_k is relatively close to $\pm R$ are called u's and v's according to the conditions

$$r_v < -R + \rho, \quad \text{and} \quad r_u > R - \rho^2, \quad \text{respectively.} \tag{3.5}$$

One may similarly introduce numbers u^* and $v^* \leq n$ by the conditions

$$r_{v^*} < -R + \rho^3, \quad \text{and} \quad r_{u^*} > R - \rho^4, \quad \text{respectively.} \tag{3.6}$$

In this section it is shown that there are many u's and many v's. We also obtain results on the distribution of the u's, v's, u^*'s and v^*'s.

By the preceding n is a u. Throughout the following we assume inequality (3.1).

Step 4. *With large n as in (3.4), the number of v's ($< n$) is at least $\frac{1}{2+\varepsilon}n - o(n)$ and the largest v is at least $n - o(n)$.*

Proof. If $k < n$ is a v one has $R + r_k < \rho$. With $q = n$, $D = \rho$ and $\eta = 1$, the fundamental relation with $\gamma = 0$ thus shows that

$$\sum_{k \text{ is a } v} a_{n-k}(1 + \text{neg}) + \sum_{k \text{ not a } v} a_{n-k} \cdot \text{neg} = n + \left(1 - \frac{R}{\rho}\right) R + o(n).$$

Since we have condition (1.1) we may apply Theorem 1.2 with $0<\gamma\leq 1/2$ to conclude that $R^2/\rho=o(n)$. It follows that

$$\sum_{k \text{ is a } v} a_{n-k} \geq n+o(n) \quad [\text{in fact,} \quad = n+o(n)].$$

Hence by (3.1) the number of v's is at least $\frac{1}{2+\varepsilon}n-o(n)$.

To prove the second statement, suppose that $\max v < \theta n$ for some $\theta \in (0,1)$. Then $R+r_k\geq\rho$ for $k\geq\theta n$, hence by the fundamental relation with q, D and η as above,

$$\sum_{k\geq\theta n} a_{n-k}\cdot \text{neg} + \sum_{k<\theta n} a_{n-k}(1+\text{neg}) = n+o(n).$$

This would imply that

$$s_n - s_{(1-\theta)n} = \sum_{k<\theta n} a_{n-k} - \mathcal{O}(1) \geq n-o(n),$$

in contradiction to Step 1.

Step 5. *There are also many u's $<n$. Indeed, let q be a (large) v, so that $R+r_q<\rho$. Then the number of u's $<q$ is at least $\frac{1}{2+\varepsilon}q-o(q)$ and the largest $u<q$ is at least $q-o(q)$.*

Proof. If k is a u one has $R-r_k<\rho^2$. With $D=\rho^2$ and $\eta=-1$, the fundamental relation with $\gamma=0$ now shows that

$$\begin{aligned}\sum_{k \text{ is a } u} a_{q-k}(1+\text{neg}) &+ \sum_{k \text{ not a } u} a_{q-k}\cdot\text{neg} \\ &= q\left(1-\frac{\lambda}{\rho}\right)+\left(1-\frac{R}{\rho^2}\right)(-R+\lambda\rho)+o(q)\end{aligned}$$

with $0\leq\lambda<1$. Thus

$$\sum_{k \text{ is a } u} a_{q-k} \geq q+\left(\frac{R}{\rho^2}-1\right)(R-\rho)-o(q),$$

while

$$\sum a_{q-k} = q+r_q < q-R+\rho.$$

It follows that

$$\frac{R^2}{\rho^2}=o(q) \quad \text{and} \quad \sum_{k \text{ is a } u} a_{q-k} \geq q-o(q).$$

Hence the number of u's less than q is at least $\frac{1}{2+\varepsilon}q-o(q)$.

For the proof that the largest $u<q$ is at least $q-o(q)$, cf. the proof in Step 4.

Remark 3.1. If $r_n < 0$ one may interchange ρ and ρ^2 in the definition of u's and v's (3.5). For the v-number n the argument of Step 4 (but with $\eta = -1$) would then give a good result on the u-numbers. One would not need Theorem 1.2 for this, cf. the proof in Step 5. Similarly, the argument of Step 5 (but with $\eta = 1$) would now give a good result on the v-numbers.

By Step 4 one may take q in Step 5 of the form $n - o(n)$. One thus has the following

Proposition 3.2. *Assuming (3.1) and (3.3) as we may, the total number of u's and v's up to n is at least $\frac{2}{2+\varepsilon}n - o(n)$.*

The method of Steps 4 and 5 also proves the following

Proposition 3.3. *If q is a "large" u – larger than $n/\log n$, say – the number of $v^* < q$ is at least $\frac{q}{2+\varepsilon}$ and the largest is at least $q - o(q)$. Similarly, if q is a large v^*, the number of $u^* < q$ is at least $\frac{q}{2+\varepsilon}$ and the largest is at least $q - o(q)$.*

3.3. A contradiction. Continuing under assumption (3.1) where one may take $\varepsilon \le \frac{1}{10}$ and under supposition (3.3) on $R_n = R$, we use Siegel's ideas to show that there are many integers $< n$ which are neither u's nor v's, enough to give a contradiction.

Step 6. *Set $L = 5R$, let t be an integer such that $\frac{n}{\log n} + L \le t \le n$ and let I_t denote the interval $t - L < q \le t$.*

(i) If I_t contains at least one u (or at least one v, respectively), then

$$r_q < R/2 \ \ (\text{or} \ \ r_q > -R/2, \ \text{resp.}) \ \ \text{for some} \ \ q \in I_t. \tag{3.7}$$

(ii) In any case the number of integers in I_t which are neither u's nor v's is at least equal to

$$R/2(1 + \varepsilon) - o(R). \tag{3.8}$$

Remark 3.4. The constant 5 in $L = 5R$ and the denominator 2 in (3.7) have been chosen experimentally to ensure that for small ε, there are enough non-u's, non-v's both here and in Proposition 3.5 below.

Proof. If I_t is free of u's and v's there is nothing to prove for Step 6. Also, the proof of parts (i) and (ii) is easy if I_t contains both a u and a v: $r_u = R - o(R)$ and $r_v = -R + o(R)$, while $|r_k - r_{k-1}| = |a_k - 1| < 1 + \varepsilon$ for $k > k_0$.

From here on we assume that I_t contains at least one u but no v (the proof is similar if I_t contains a v but no u).

(α) Suppose now that (3.7) is false, that is, $r_q \ge R/2$ throughout I_t. Then we can show that there are too many non-u^*'s up to t. For the u^*'s among the k's $\le t$, one has $R+r_k > 2R-\rho^4$. Hence by the fundamental relation with $D=2R-\rho^4$ and $\eta=1$,

$$\sum_{k \text{ is a } u^*} a_{q-k}\cdot \text{neg} + \sum_{k \text{ not a } u^*} a_{q-k}(1+\text{neg}) = q\left(1-\frac{R-r_q}{2R-\rho^4}\right)+o(q).$$

We set $a_\nu = 0$ and $s_\nu = 0$ for $\nu < 1$. Then since $R - r_q \le R/2$,

$$\sum_{k\le t,\, k \text{ is not } u^*} a_{q-k} = \sum_{k<q,\, k \text{ is not } u^*} a_{q-k} \ge \frac{3}{4}q - o(q) \tag{3.9}$$

whenever $t-L < q \le t$. Defining $\chi(k)=1$ if $k\ge 1$ is a non-u^* and $\chi(k)=0$ otherwise, we sum over q. Then the left-hand side of (3.9) gives the sum

$$\sum_{t-L<q\le t}\ \sum_{k\le t}\chi(k)a_{q-k} = \sum_{k\le t}\chi(k)\sum_{t-L<q\le t} a_{q-k}.$$

Now the final inner sum is equal to $s_{t-k}-s_{t-k-L} \le L+2R$, so that the repeated sum on the left is $\le (L+2R)\sum_{k\le t}\chi(k)$. Also summing over q in the right-hand side of (3.9) one thus obtains the inequality

$$(L+2R)\sum_{k\le t}\chi(k) \ge \frac{3}{4}Lt - o(Lt) \ \text{ or } \sum_{k\le t,\, k \text{ is not } u^*} 1 \ge \frac{15}{28}t - o(t). \tag{3.10}$$

Recall that the given interval I_t contains a u, hence if we use both parts of Proposition 3.3, one after the other, we find that the number of $u^* \le t$ is at least $\frac{t}{2+\varepsilon} - o(t)$. But since $\varepsilon \le \frac{1}{10}$ this is inconsistent with (3.10):

$$\frac{15}{28}+\frac{1}{2+\varepsilon} \ge \frac{85}{84} > 1.$$

The contradiction establishes the relevant part of (3.7).

(β) We continue under the assumption that I_t contains a u but no v. For part (ii) we now choose a $q\in I_t$ for which $r_q < R/2$. Let u' be the (or a) u in I_t closest to q; there will be no u's (and no v's) between u' and q. We have $r_{u'} > R-\rho^2$, hence $|r_{u'}-r_q| > R/2 - o(R)$. Since $|r_{u'}-r_q| < (1+\varepsilon)|u'-q|$, it follows that $|u'-q| > R/2(1+\varepsilon) - o(R)$. Thus the total number of non-u's, non-v's in I_t is at least $R/2(1+\varepsilon) - o(R)$.

Step 7. It remains to sum over appropriate intervals I_t.

Proposition 3.5. *Let the a_k's satisfy (3.1) with $\varepsilon \le \frac{1}{10}$, let $R = R_n$ be unbounded and let n be as in (3.3). Then the number of integers $\le n$ which are neither u's nor v's is at least*

$$\frac{1}{10(1+\varepsilon)}n - o(n). \tag{3.11}$$

Proof. We continue with the notation of Step 6 so that in particular $\frac{n}{\log n}+L\le t\le n$. Observe that the intervals $I_t=(t-L,t]$ belong to this range for $t=n,n-L,\dots,t-(N-1)L$ where $NL\approx n-\frac{n}{\log n}$. Summing over these intervals I_t, we conclude from part (ii) in Step 6 that the total number of non-u's, non-v's $\le n$ is at least

$$\left\{\frac{1}{2(1+\varepsilon)}R-o(R)\right\}\frac{NL}{L}\ge\left\{\frac{1}{2(1+\varepsilon)}R-o(R)\right\}\frac{n-o(n)}{5R}$$
$$=\frac{1}{10(1+\varepsilon)}n-o(n).$$

Conclusion 3.6. *For $\varepsilon\le\frac{1}{10}$, Proposition 3.5 contradicts Proposition 3.2. This contradiction shows that supposition (3.3) is false, in other words,*

$$r_n=s_n-n=\mathcal{O}(1).$$

Indeed, for $\varepsilon\le\frac{1}{10}$,

$$\frac{2}{2+\varepsilon}+\frac{1}{10(1+\varepsilon)}\ge\frac{20}{21}+\frac{1}{11}>1.$$

Hence if (3.3) would be true, Propositions 3.2 and 3.5 would show that for large n, the total number of u's, v's and non-u's, non-v's $\le n$ would exceed n.

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